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ON THE LUNAR INEQUALITIES PRODUCED BY THE MOTION OF THE ECLIPTIC.

By Dr. G. W. Hill, Washington, D. C.

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II.

The direct action of the planets produces in the motion of the moon terms which have nearly the same periods as those we have been considering. To complete the subject it is necessary to derive these and add them to those just obtained. If $\delta' R$ denote the part of R which is due to the action of the planet m'', and Δ the distance of the latter from the earth, two accents being used to denote quantities which belong to the planet,

$$\delta' R = m'' \left\{ \left[\Delta^2 - 2\,(x''x + y''y + z''z) + r^2\right]^{-\frac{1}{2}} - \frac{x''x + y''y + z''z}{r''^3} \right\}.$$

Or, with sufficient approximation,

$$\delta' R = \frac{m''}{2} \left\{ 3 \frac{(x''x + y''y + z''z)^2}{\Delta^5} - \frac{r^2}{\Delta^3} \right\}.$$

The only part of $\delta' R$ which can produce terms we are in search of is that which has z'' for a factor: thus we may take

$$\delta' R = 3m'' \frac{(x''x + y''y)\,z''z}{\Delta^5}.$$

But, with sufficient approximation,

$$\begin{aligned} x'' &= a'' \cos(\varepsilon'' + n''t) + a' \cos(\varepsilon' + n't), \\ y'' &= a'' \sin(\varepsilon'' + n''t) + a' \sin(\varepsilon' + n't), \\ z'' &= a''\gamma'' \sin(\varepsilon'' + n''t - \Omega''), \\ \Delta^{-5} &= \frac{1}{2} A_0 + A_1 \cos\left[\varepsilon'' - \varepsilon' + (n'' - n')\,t\right] + \dots, \\ x''x + y''y &= a''r \cos(\lambda - \varepsilon'' - n''t) + a'r \cos(\lambda - \varepsilon' - n't). \end{aligned}$$

Preserving only terms which are needed,

$$\begin{aligned} (x''x + y''y)\,z'' = \frac{1}{2} a''\gamma'' r \Big\{ & a'' \sin(\lambda - \Omega'') \\ & + a' \sin\left[\lambda - \Omega'' + \varepsilon'' - \varepsilon' + (n'' - n')\,t\right] \Big\}, \end{aligned}$$

$$\frac{(x''x + y''y)\,z''z}{\Delta^5} = \frac{1}{4} a''\gamma''(a''A_0 + a'A_1)\, rz \sin(\lambda - \Omega'').$$

Consequently

$$
\begin{aligned}
\delta' R &= \frac{3}{4} m'' a'' \gamma'' (a'' A_0 + a' A_1)\, rz \sin(\lambda - \Omega'') \\
&= \frac{3}{4} \frac{m''}{m'}\, m^2 a'^3 a'' \gamma''\, (a'' A_0 + a' A_1)\, rz \sin(\lambda - \Omega'') \\
&= -\, 2\, Krz \sin(\lambda - \Omega'')\,.
\end{aligned}
$$

For an inferior planet

$$A_0 = a'^{-5}\, b^{(0)}_{\frac{5}{2}}, \qquad A_1 = -\, a'^{-5}\, b^{(1)}_{\frac{5}{2}},$$

and for a superior one

$$A_0 = a''^{-5}\, b^{(0)}_{\frac{5}{2}}, \qquad A_1 = -\, a''^{-5}\, b^{(1)}_{\frac{5}{2}}.$$

But

$$b^{(0)}_{\frac{5}{2}} = \frac{(1 + \alpha^2)\, b^{(0)}_{\frac{3}{2}} + \frac{2}{3} \alpha b^{(1)}_{\frac{3}{2}}}{(1 - \alpha^2)^2}, \quad b^{(1)}_{\frac{5}{2}} = \frac{2 \alpha b^{(0)}_{\frac{3}{2}} + \frac{1}{3} (1 + \alpha^2)\, b^{(1)}_{\frac{3}{2}}}{(1 - \alpha^2)^2}\,.$$

Consequently, for an inferior planet,

$$K = \frac{3}{8} \frac{m''}{m'} m^2 \gamma'' \frac{\alpha}{1 - \alpha^2} \left\{ \alpha b^{(0)}_{\frac{3}{2}} + \frac{1}{3} b^{(1)}_{\frac{3}{2}} \right\},$$

and, for a superior,

$$K = -\, \frac{3}{8} \frac{m''}{m'} m^2 \gamma'' \frac{\alpha^3}{1 - \alpha^2} \left\{ b^{(0)}_{\frac{3}{2}} + \frac{1}{3} \alpha b^{(1)}_{\frac{3}{2}} \right\}.$$

To determine δz we shall have the equation

$$\frac{d^2 \delta z}{d\zeta^2} + A \delta z = -\, 2 K r \sin(\lambda - \Omega'').$$

Making $\varepsilon + nt - \Omega'' = \zeta'$,

$$K r \sin(\lambda - \Omega'') = \left\{ 1 - \frac{m^2}{6} \right\} K \sin \zeta' - \frac{19}{16} m^2 K \sin(\zeta' - 2\tau).$$

Since K is much smaller than $\frac{\pi}{n}$, we shall content ourselves with one order of approximation less in the factors which multiply it than in those which multiply $\frac{\pi}{n}$. With this restriction, it will be readily seen that the value of δz is obtained simply by writing K and ζ' respectively for $\frac{\pi}{n}$ and ζ in the formula previously obtained. Thus

$$\delta z = -\left\{ \frac{4}{3} m^{-2} + \frac{1}{2} m^{-1} + \frac{31}{9} \right\} K \sin \zeta'$$

$$+\left(\frac{1}{2}m^{-1}+\frac{25}{12}\right)K\sin(\zeta'-2\tau)$$
$$-\frac{1}{4}K\sin(\zeta'+2\tau).$$

As regards the differential equations which determine $r\delta r$ and $\delta\lambda$, it is evident that they remain the same as before, with the exceptions that K and ζ' everywhere take the place of $\frac{\pi}{n}$ and ζ; and in U', in place of $2\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$, must be put $4\delta'R=0$, and that consequently U' in this case becomes

$$U'=\gamma K\left[2\cos(\zeta'-\eta-2\tau)+2\cos(\zeta'-\eta+2\tau)-\cos(\zeta'+\eta-2\tau)\right];$$

and in U'', in place of $2\frac{\pi}{n}\int xdz$, must be put

$$-2K\int rz\cos(\lambda-\Omega'')\,d\zeta=-\frac{3}{16}\gamma K\cos(\zeta'+\eta-2\tau),$$

whence U'' in this case becomes

$$U''=\gamma K\left\{\left(\frac{1}{2}m^{-1}+\frac{1}{2}\right)\cos(\zeta'-\eta-2\tau)+\left(\frac{1}{2}m^{-1}+\frac{5}{16}\right)\cos(\zeta'-\eta+2\tau)\right.$$
$$\left.+\left(\frac{4}{3}m^{-2}+\frac{1}{2}m^{-1}\right)\cos(\zeta'+\eta)-\left(\frac{1}{2}m^{-1}+\frac{7}{3}\right)\cos(\zeta'+\eta-2\tau)\right\};$$

and, in the differential equation for determining the coefficient of sin $(\zeta-\eta)$ in $\delta\lambda$, in place of $-4\frac{\pi}{n}\frac{ydz-zdy}{d\zeta}$, must be put

$$-7\delta'R=7\gamma K\cos(\zeta'-\eta).$$

Making use of similar expressions for $r\delta r$ and $\delta\lambda$ as were used in the former case, we obtain the equations of condition

$$-3B_2+3m^2C_1=2,$$
$$-3B_3-3m^2C_1=2,$$
$$-B_5-\frac{3}{2}m^2C_4=1,$$
$$\left.\begin{aligned}&\left(\frac{3}{4}m^2-\frac{9}{32}m^3\right)C_1-\frac{9}{2}m^2(C_2-C_3)\\&-\frac{57}{2}m^3C_1+6m(B_2-B_3+C_2+C_3)\end{aligned}\right\}=-\frac{1}{2}+\frac{9}{16}m,$$
$$(2-2m)C_2-\frac{3}{4}m^2C_1-2B_2=-\frac{1}{2}m^{-1}-\frac{1}{2},$$
$$(2-2m)C_3-\frac{3}{4}m^2C_1+2B_3=\frac{1}{2}m^{-1}+\frac{5}{16},$$

$$2C_4 = \frac{4}{3}m^{-2} + \frac{1}{2}m^{-1},$$

$$2mC_5 - \left(\frac{3}{4}m + \frac{55}{32}m^2\right) C_4 + 2B_5 = -\frac{1}{2}m^{-1} - \frac{7}{3}.$$

These equations are the same as we obtained in the case of the inequalities produced by the motion of the ecliptic, with the single exception of that which determines C_1; and, being solved, they give

$$\begin{aligned}
\delta\lambda = & -\left(\frac{2}{3}m^{-2} + \frac{3}{2}m^{-1}\right) \gamma K \sin(\zeta' - \eta) \\
& - \frac{1}{4}m^{-1} \gamma K \sin(\zeta' - \eta - 2\tau) \\
& + \frac{1}{4}m^{-1} \gamma K \sin(\zeta' - \eta + 2\tau) \\
& + \left(\frac{2}{3}m^{-2} + \frac{1}{4}m^{-1}\right) \gamma K \sin(\zeta' + \eta) \\
& + \frac{3}{2}m^{-1} \gamma K \sin(\zeta' + \eta - 2\tau).
\end{aligned}$$

The expression for the inequalities in latitude is

$$\begin{aligned}
\delta\beta = & -\left(\frac{4}{3}m^{-2} + \frac{1}{2}m^{-1} + \frac{11}{3}\right) K \sin \zeta' \\
& + \left(\frac{1}{2}m^{-1} + \frac{17}{12}\right) K \sin(\zeta' - 2\tau) \\
& - \frac{11}{12} K \sin(\zeta' + 2\tau).
\end{aligned}$$

III.

It remains only to transform the foregoing formulæ into numerical results According to Hansen and Olufsen (*Tables du Soleil, Introduction*),

$$\pi \sin \Pi = +\, 0''.053916, \qquad \pi \cos \Pi = -\, 0''.467839,$$

whence

$$\pi = 0''.470903, \text{ and } \Pi = 173^\circ 25' 34''.$$

Also

$$\begin{aligned}
n &= 17325225'', \\
m &= 0.074801, \\
\gamma &= 0.089673, \\
e &= 0.054731.
\end{aligned}$$

Substituting these values, the inequalities produced by the motion of the ecliptic are

$$\begin{aligned}
\delta\lambda = &+ 0''.2952 \sin(\zeta - \eta) + 0''.0000 \sin(\zeta - \eta - 2\tau) \\
&+ 0''.0045 \sin(\zeta - \eta + 2\tau) + 0''.0616 \sin(\zeta + \eta) \\
&+ 0''.0089 \sin(\zeta + \eta - 2\tau) + 0''.0004 \sin(\zeta + \eta + 2\tau) \\
&+ 0''.0000 \sin(\zeta + \eta - 4\tau), \\
\delta\beta = &- 1''.4001 \sin\zeta + 0''.0469 \sin(\zeta - 2\tau) \\
&- 0''.0064 \sin(\zeta + 2\tau) + 0''.0001 \sin(\zeta - 4\tau) \\
&+ 0''.0757 \sin(\zeta - \xi) - 0''.0768 \sin(\zeta + \xi) \\
&+ 0''.0088 \sin(\zeta - 2\tau + \xi) - 0''.0137 \sin(\zeta + 2\tau - \xi) \\
&+ 0''.0022 \sin(\zeta - 2\tau - \xi) - 0''.0007 \sin(\zeta + 2\tau + \xi) \\
&+ 0''.0003 \sin(\zeta - 4\tau + \xi).
\end{aligned}$$

To compute the terms due to the direct action of the planets we take for Venus,

$$\frac{m''}{m'} = \frac{1}{408134}, \qquad \gamma'' = \tan(3^\circ\, 23'\, 34''), \qquad \Omega'' = 75^\circ\, 21',$$

for Mars,

$$\frac{m''}{m'} = \frac{1}{3200900}, \qquad \gamma'' = \tan(1^\circ\, 51'), \qquad \Omega'' = 48^\circ\, 24',$$

for Jupiter,

$$\frac{m''}{m} = \frac{1}{1050}, \qquad \gamma'' = \tan(1^\circ\, 18'\, 35''), \qquad \Omega'' = 98^\circ\, 57',$$

for Saturn,

$$\frac{m''}{m} = \frac{1}{3512}, \qquad \gamma'' = \tan(2^\circ\, 29'), \qquad \Omega'' = 112^\circ\, 21'.$$

The quantities depending on the ratio of the mean distances are taken from Runkle's *Tables of the Coefficients of the Perturbative Function.* Thus we obtain for the several planets, in their order, the values of log K expressed in seconds of arc: —

$$\begin{aligned}
\log K &= 96.9867, \\
\log K &= 95.2450_n, \\
\log K &= 96.1878_n, \\
\log K &= 95.1081_n.
\end{aligned}$$

Then the action of Venus produces the following terms: —

$$\begin{aligned}
\delta\lambda &= - 0''.0121 \sin(\zeta' - \eta) + 0''.0106 \sin(\zeta' + \eta), \\
\delta\beta &= - 0''.2412 \sin\zeta' \qquad + 0''.0078 \sin(\zeta' - 2\tau).
\end{aligned}$$

The action of Mars produces the terms

$$\begin{aligned}\delta\lambda &= +\,0''.0003 \sin(\zeta'' - \eta) - 0''.0002 \sin(\zeta' + \eta),\\ \delta\beta &= +\,0''.0044 \sin \zeta''\end{aligned}$$

The action of Jupiter produces the terms

$$\begin{aligned}\delta\lambda &= +\,0''.0019 \sin(\zeta''' - \eta) - 0''.0016 \sin(\zeta''' + \eta),\\ \delta\beta &= +\,0''.0383 \sin \zeta''' \qquad - 0''.0012 \sin(\zeta''' - 2\tau).\end{aligned}$$

The action of Saturn produces the terms

$$\begin{aligned}\delta\lambda &= +\,0''.0002 \sin(\zeta^{\text{iv}} - \eta) - 0''.0001 \sin(\zeta^{\text{iv}} + \eta),\\ \delta\beta &= +\,0''.0031 \sin \zeta^{\text{iv}}.\end{aligned}$$

The terms having the same period in the indirect and direct actions of the planets may be united in a single term, and we have

$$\begin{aligned}\zeta - \eta &= \Omega - \Pi + 90^\circ, &\quad \zeta + \eta &= 2\text{☾} - \Omega - \Pi + 90^\circ,\\ \zeta' - \eta &= \Omega - \Omega'', &\quad \zeta' + \eta &= 2\text{☾} - \Omega - \Omega''.\end{aligned}$$

Thus, preserving only the terms whose coefficients exceed $0''01$, the value of $\delta\lambda$ due to both the indirect and direct action of the planets, is

$$\begin{aligned}\delta\lambda &= +\,0''.0305 \sin \Omega - 0''.2838 \cos \Omega\\ &\quad +\,0''.0100 \sin(2\text{☾} - \Omega) - 0''.0697 \cos(2\text{☾} - \Omega).\\ &= +\,0''.2854 \sin(\Omega + 276^\circ\ 8') + 0''.0704 \sin(2\text{☾} - \Omega + 278^\circ),\end{aligned}$$

In the case of the latitude we may write the true orbit longitude L of the moon in the place of the mean in the principal term and neglect the remaining terms. Thus the value of $\delta\beta$, due to both actions of the planets, is

$$\begin{aligned}\delta\beta &= -\,0''.2256 \sin L + 1''.5802 \cos L\\ &= +\,1''.5963 \sin(L + 98^\circ\ 8'.6).\end{aligned}$$

The terms in $\delta\lambda$ and $\delta\beta$ which involve $\sin \Omega$ and $\sin L$ coalesce with the principal inequalities which are due to the figure of the earth and have the same arguments. Hansen (*Tables de la Lune*, pp. 8, 15) has, respectively in the perturbed mean anomaly and latitude, the terms $+\,7''.760 \sin(184^\circ\ 42' - \Omega)$ and $+\,8''.764 \sin(L + 169^\circ\ 51')$. The parts of these which depend on $\cos \Omega$ and $\cos L$ are $-\,0''.636 \cos \Omega$ and $+\,1''.544 \cos L$. In the *Darlegung* he gives coefficients somewhat different. As to $\delta\beta$, Hansen's value nearly coincides with mine, but his coefficient in $\delta\lambda$ is more than double mine. This discrepancy is probably to be attributed to difference of the system of co-ordinates employed.

The values of these terms which Sir G. B. Airy has determined from observation, in his first memoir on the correction of the lunar elements (*Mem. Astr Soc.*, Vol. XVII), are

$$\delta\lambda = -0''.97 \cos \Omega, \qquad \delta\beta = +2''.17 \cos L.$$

These he has changed to

$$\delta\lambda = -1''.06 \cos \Omega, \qquad \delta\beta = +1''.93 \cos L,$$

in his second memoir (*Mem. Astr. Soc.*, Vol. XXIX).

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SYLVESTER'S TERNARY CUBIC FORM.

By Mr. A. M. SAWIN, Madison, Wis.

Having noticed with interest some of the deductions of Professor Sylvester from what he denominates his Ternary Cubic,* I desire to add a few observations upon this subject and to call attention to some peculiar properties of this cubic which I believe have not been alluded to. The ternary cubic $x^3 - 3xy^2 \pm y^3$ is described by Professor Sylvester† as exhibiting the property, that, if any prime number of the form $18n \pm 1$ or $3(18n \pm 1)$, where n is zero or some integer, can be represented by integer values of x, y in the above expression, then is that number or its nonuple the sum of two cubes. As a matter of fact all such numbers below 537 inclusive (39 in all) were ascertained by Professor Sylvester to be thus representable by integer values of x, y, which he presumed according to the usual canons of induction were sufficient to generalize this remarkable property of the above ternary cubic form to all numbers.‡ Although these actual verifications, are far from establish the generality of the expression for all such numbers; nevertheless, we shall see that this cubic does possess some interesting general properties.

THEOREM I.

If in the equation $x^3 - 3xy^2 \pm y^3 = 18n \pm 1$ or $3(18n \pm 1)$, x and y be any two numbers prime to each other, then will n be zero or some integer.

LEMMA I.

The difference of any two numbers prime to each other is prime to either.

Let x and y represent any two numbers prime to each other; and put x

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***American Journal of Mathematics*, Vol. II. p. 280.

† The same, Vol. II. p. 283.

‡ The same, Vol. III. p. 58.